Technische Universität München, Department of Mathematics
Chair of Operations Research

Special Topics in Algorithmic Game Theory (MA5226)

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Exercise Sheet 4 - Due Wednesday, May 9

Hand in your answers, either in person or by e-mail to `diogo.pocas@tum.de` *(preferably as a single file).* ***No late submissions accepted.***

Exercise 4.1 (Pareto distribution, related to Exercise 5.6 from [20LAGT])
Consider a single-bidder, single-item auction; the bidder's valuation is drawn from the distribution

$$F(v) = 1 - \frac{1}{v} \quad \text{for } v \in [1, \infty).$$

Verify that this distribution is regular (cf. Exercise 2.4(c) from last week). Consider a DSIC mechanism that sets some selling price $R \geq 1$. Argue that the expected revenue does *not* necessarily equal its expected virtual welfare. How do you reconcile this observation with Theorem 5.2 from [20LAGT]?

Exercise 4.2 (Computation of Pigou bounds, Exercises 11.1 and 11.2 from [20LAGT])

(a) Prove that if $\mathcal{C}$ is the set of cost functions of the form $c(x) = ax + b$ with $a, b \geq 0$, then the Pigou bound $\alpha(\mathcal{C}) = \frac{4}{3}$.

(b) Prove that if $\mathcal{C}$ is the set of nonnegative, nondecreasing, and concave cost functions, then the Pigou bound $\alpha(\mathcal{C}) = \frac{4}{3}$.

Exercise 4.3 (Pigou network with monomial cost function)
For any positive integer p, consider the (nonlinear) Pigou network with one unit of incoming traffic and cost functions $c_1(x) = 1$, $c_2(x) = x^p$. Show that the Price of Anarchy is $\Omega\left(\frac{p}{\log p}\right)$.

(*Hint: you can use without proof the approximation* $1 \leq \sqrt[p]{p+1} \leq 1 + 2\frac{\ln(p+1)}{p}$, *valid for any* $p \geq 1$.)

Exercise 4.4 (Multicommodity networks, Exercise 11.5 from [20LAGT])
Consider a *multicommodity* network $G = (V, E)$, where for each $i = 1, 2, \ldots, k$, r_i units of traffic travel from an origin $o_i \in V$ to a destination $d_i \in V$.

(a) Extend the definitions of a flow and of an equilibrium flow (Definition 11.3 from [20LAGT]) to multicommodity networks.

(b) Extend the two expressions (11.3) and (11.4) from [20LAGT] for the total travel time to multicommodity networks.

(c) Prove that Theorem 11.2 from [20LAGT] continues to hold for multicommodity networks.

Exercise 4.5 (Alternative threshold choice for prophet inequality)
Consider a sequence $G_1, \ldots, G_n$ of independent distributions as in the statement of the prophet inequality. Show that setting a threshold of $t^* = \frac{1}{2}\,\mathbf{E}_{\pi\sim\mathbf{G}}\left[\max_i \pi_i\right]$ (and keeping the first reward that meets his threshold) is a strategy that also yields expected reward at least $\frac{1}{2}\,\mathbf{E}_{\pi\sim\mathbf{G}}\left[\max_i \pi_i\right]$.

This problem set will be discussed in the tutorials on May 11th, 2018.