Special Topics in Algorithmic Game Theory
Lecture 3 – Supplementary Notes

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1 Myerson's Lemma

Theorem 1 (Myerson's Lemma [2]). *A (direct-revelation) mechanism* $(\mathbf{x}, \mathbf{p})$ *is DSIC, if and only if*

1. *its allocation rule* $\mathbf{x}$ *is monotone, and*
2. *the payment charged to player* i *is given by the formula*

$$p_i(\mathbf{b}) = b_i \cdot x_i(\mathbf{b}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i})\, dz. \tag{1}$$

Proof. ($\Longrightarrow$) Fix some DSIC mechanism $(\mathbf{x}, \mathbf{p})$. Also fix some bidder i, and all other players' bids $\mathbf{v}_{-i}$. For simplicity, let's denote $x(z) \equiv x_i(z, \mathbf{v}_{-i})$ and $p(z) \equiv p_i(z, \mathbf{v}_{-i})$. Also, denote with $u(z)$ the utility of player i when she truthfully bids z, i.e.,

$$u(z) = x(z) \cdot z - p(z). \tag{2}$$

Then, due to DSIC, it must be that

$$x(z)z - p(z) = u(z) \geq x(y)z - p(y) \tag{3}$$
$$x(y)y - p(y) = u(y) \geq x(z)y - p(z). \tag{4}$$

The first inequality simply corresponds to the fact that when player i has true value z, his utility is always maximized when truthfully reporting z, instead of any other value y. The second inequality is corresponds to the symmetric situation, by switching variables z and y. Subtracting (3) from (4) we get that

$$[x(z)y - p(z)] - [x(z)z - p(z)] \leq u(y) - u(z) \leq [x(y)y - p(y)] - [x(y)z - p(y)],$$

or equivalently,

$$x(z)(y - z) \leq u(y) - u(z) \leq x(y)(y - z). \tag{5}$$

From the above inequality (5), just by keeping the left- and right-hand side quantities, we can see that

$$z < y \Longrightarrow x(z)(y - z) \leq x(y)(y - z) \Longrightarrow x(z) \leq x(y),$$

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proving that indeed $x_i(z, \mathbf{v}_{-i})$ is nondecreasing with respect to player i's report z.

To show now that the payment rule $p(z)$ is given by (1), observe once more (5): it says that[1] $x(z)$ is a *subgradient*[2] of function u at z, and this holds for all possible bids z. As a result, the utility function $u(z)$ is an absolutely continuous function, with the Fundamental Theorem of Calculus giving that

$$u(b) = u(0) + \int_0^b x(z)\, dz. \tag{6}$$

Recall that, we require our mechanisms to be *Individual Rational (IR)*, that is, $u(z) \geq 0$ for all z; and furthermore, that $p(z) \geq 0$ for all z (*No Positive Transfers*).[3] Thus, it must that $u(0) = x(0) \cdot 0 - p(0) = -p(0) = 0$, and so (6) finally becomes

$$x(b)b - p(b) = \int_0^b x(z)\, dz,$$

which is exactly (1).

($\Longleftarrow$) For the inverse direction now, assume that an allocation rule $\mathbf{x}$ is monotone and that its payment rule $\mathbf{p}$ is given by (1). We will show that mechanism $(\mathbf{x}, \mathbf{p})$ is DSIC. Fix some player i and the bids of all other players $\mathbf{b}_{-i}$. Also fix a *true* valuation $v \equiv v_i$ for player i. Again, for simplicity, we drop the subscripts i from now on. The utility of player i when she reports bid b is

$$x(b) \cdot v - p(b) = x(b)v - \left[x(b)b - \int_0^b x(z)\, dz\right] = x(b) \cdot (v - b) + \int_0^b x(z)\, dz.$$

And when she truthfully reports $b = v$, it is

$$x(v) \cdot v - p(v) = \int_0^v x(z)\, dz = \int_0^b x(z)\, dz + \int_b^v x(z)\, dz \geq \int_0^b x(z)\, dz + x(b) \cdot (v - b),$$

the last inequality holding due to the fact that $x(z)$ is nondecreasing. Thus, misreporting a general bid b that might not be equal to v, can only give you less utility, proving the DSIC property.

See also Figure 14.6 in Karlin and Peres [1] for a nice graphical representation of this proof. □

Your textbook [20LAGT] gives two alternative formulas for the payment rule (1) above, under further special assumptions on the allocation function $x_i(z) \equiv x_i(z, \mathbf{b}_{-i})$ of player i.

In general, we only know that this function is nondecreasing with respect to player i's bid z (keeping everything else $\mathbf{b}_{-i}$ fixed), due to Property 1 of Theorem 1. However, by assuming that it is additionally *piecewise constant*[4], we get that

$$p_i(\mathbf{b}) = \sum_{j=1}^{\ell} z_j \cdot [\text{jump of } x_i(\cdot, \mathbf{b}_{-i}) \text{ at } z_j], \tag{7}$$

[1] As a matter of fact, just the left-hand side inequality is enough for this.

[2] For more details see,e.g., Proposition (11) in Karlin and Peres [1, Appendix C].Slightly sacrificing rigorousness here, you can think of this in the following way: taking any $y > z$ and dividing (5) by $y - z$, you get $x(z) \leq \frac{u(y)-u(z)}{y-z} \leq x(y)$. Taking the limit $y \to z$, you can see that $x(z)$ is equal to the derivative of u at z; the allocation rule is the derivative of the player's utility! However, $x(z)$ might not be continuous at all points z, and thus, $u(z)$ might not be differentiable. Nevertheless, since $x(z)$ is monotone, we know that such "problematic" behaviour can only occur on a set of points with zero (Lebesgue) measure.

[3] See Eq. (3.1) in your textbook [20LAGT].

[4] This is the case, e.g., with the single-item and sponsored search auction examples we have seen before in class.

where $z_1, z_2, \dots, z_\ell$ are the breakpoints of the allocation function $x_i(z)$ as player's i bid z ranges within $[0, b_i]$.

Alternatively, if we know that $x_i(z)$ is *continuously differentiable*[5] (with derivative $x_i'(z) = \frac{\partial x_i(z,\mathbf{b}_{-i})}{\partial z}$), then we have that

$$p_i(\mathbf{b}) = \int_0^{b_i} z \cdot x_i'(z)\, dz. \tag{8}$$

It is not difficult to see that (7) and (8) are special cases of the general formula (1). Indeed, first for the case where $x_i(z)$ is piecewise constant, observe that the quantity at the right-hand side of (7) is just the area *above* the graph of $x_i(z)$ *inside* the box $[0, b_i] \times [0, x_i(b_i)]$ of the Euclidean plane $\mathbb{R}^2$; thus, it must equal the total area of that box, which is $b_i \cdot x_i(b_i)$, *minus* the area *below* the graph, which is $\int_0^{b_i} x_i(z)\, dz$. This difference, is exactly the right-hand part of (1).

Next, for the case where $x_i(z)$ is continuously differentiable, by using integration by parts, we can rewrite the integral in the right-hand part of (8) as

$$\begin{aligned}
\int_0^{b_i} z \cdot x_i'(z)\, dz &= [z \cdot x_i(z)]_{z=0}^{z=b_i} - \int_0^{b_i} (z)' \cdot x_i(z)\, dz \\
&= b_i \cdot x_i(b_i) - 0 \cdot x_i(0) - \int_0^{b_i} x_i(z)\, dz \\
&= b_i \cdot x_i(b_i) - \int_0^{b_i} x_i(z)\, dz,
\end{aligned}$$

which is exactly the right-hand side of (1).

References

[1] A. R. Karlin and Y. Peres. *Game Theory, Alive.* American Mathematical Society, 2017. ISBN 9781470419820. URL https://homes.cs.washington.edu/~karlin/GameTheoryBook.pdf.

[2] R. B. Myerson. Optimal Auction Design. *Mathematics of Operations Research*, 6(1):58–73, 1981. doi: 10.1287/moor.6.1.58.

[5]As a matter of fact, a weaker requirement on $x_i(z)$ is sufficient here, namely that of *absolute continuity*.