Special Topics in Algorithmic Game Theory
Lecture 8 – Supplementary Notes

Yiannis Giannakopoulos*

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1 Potential Games

Definition 1 (Potential Game)**.** A (finite) cost-minimization[1] game $\mathcal{G} = (N, \{S_i\}_{i\in N}, \{C_i\}_{i\in N})$ is a *potential game*, if there exists a function $\Phi : \mathbf{S} \longrightarrow \mathbb{R}$ such that

$$C_i(s_i', \mathbf{s}_{-i}) - C_i(\mathbf{s}) = \Phi(s_i', \mathbf{s}_{-i}) - \Phi(\mathbf{s}), \tag{1}$$

for any strategy profile $\mathbf{s} \in \mathbf{S} = \times_{i\in N} S_i$, any player $i \in N$, and any strategy $s_i' \in S_i$. Such a function Φ, that satisfies (1), is called a *potential* of game $\mathcal{G}$.

Example 1. The Prisoner's Dilemma game (see Table 1) that we discussed in Lecture 1, is a potential game. A potential function for it can be seen in Table 2.

		Prisoner 2	
		confess	*silent*
Prisoner 1	*confess*	5, 5	1, 10
	silent	10, 1	2, 2

Table 1: Prisoner's Dilemma

		Prisoner 2	
		confess	*silent*
Prisoner 1	*confess*	0	5
	silent	5	6

Table 2: A potential function for Prisoner's Dilemma

Theorem 1. *Every potential game has at least one pure Nash equilibrium.*

*Chair of Operations Research, Department of Mathematics, TU Munich. Email: yiannis.giannakopoulos@tum.de

[1]The definition would be identical for utility-maximization games: just replace C_i by u_i.

Proof. Fix a (finite, cost-minimization) potential game $\mathcal{G} = (N, \{S_i\}_{i \in N}, \{C_i\}_{i \in N})$, along with some potential function Φ for $\mathcal{G}$. Let

$$\mathbf{s}^* \in \operatorname*{argmin}_{\mathbf{s} \in \mathbf{S}} \Phi(\mathbf{s})$$

be a minimizer of the potential function; this is well-defined, since the space $\mathbf{S}$ of all feasible strategy profiles is finite. We will prove that $\mathbf{s}^*$ is a pure Nash equilibrium of $\mathcal{G}$.

Indeed, for any player i and all strategies $s_i' \in S_i$ we have

$$C_i(s_i', \mathbf{s}_{-i}^*) - C_i(\mathbf{s}^*) = \Phi(s_i', \mathbf{s}_{-i}^*) - \Phi(\mathbf{s}^*) \geq 0, \tag{2}$$

which is equivalent to the pure Nash equilibrium condition

$$C_i(\mathbf{s}^*) \leq C_i(s_i', \mathbf{s}_{-i}^*).$$

The first equality in (2) holds because Φ is a potential, and the last inequality because $\mathbf{s}^*$ is a minimizer of Φ. □

2 Congestion Games

A *congestion game* consists of

- A finite set of *resources* (or *facilities*) E. Each resource $e \in E$ has a cost function $c_e : \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0}$, representing the *congestion* induced on it, as a function of the number of players that use it.
- A (finite) set of players $i = 1, 2, \dots, n$, the strategies of whom are sets of resources. Formally, the strategy set of each player i is $S_i \subseteq 2^E$.
- The cost of player i on a given strategy profile $\mathbf{s} \in \mathbf{S} = S_1 \times \dots S_n$ is the total congestion experienced by i (on her set s_i of selected resources), that is

  $$C_i(\mathbf{s}) = \sum_{e \in s_i} c_e(f_e(\mathbf{s})), \tag{3}$$

  where $f_e(\mathbf{s}) = |\{i \mid e \in s_i\}|$ denotes the number of players that use resource e in $\mathbf{s}$.

Comparing this definition with that of atomic selfish routing games of the previous lecture, we can immediately see that congestion games are a generalization of routing games: the edges of the routing graph G are the resources, and origin-destination $o_i \to d_i$ paths are just (a particular kind of) sets of edges. So, we have the following:

Remark 1. *Every atomic selfish routing game is a congestion game.*

Similarly to routing games, the *social cost* of a strategy profile $\mathbf{s}$ of a congestion game is the total congestion experienced by all players, i.e.

$$C(\mathbf{s}) = \sum_{i=1}^{n} C_i(\mathbf{s}) = \sum_{i=1}^{n} \sum_{e \in s_i} c_e(f_e(\mathbf{s})) = \sum_{e \in E} f_e(\mathbf{s}) \cdot c_e(f_e(\mathbf{s})). \tag{4}$$

Theorem 2. *Every congestion game*[2] *has at least one pure Nash equilibrium.*

[2] And thus, by to Remark 1, also every atomic selfish routing game.

Proof. Due to Theorem 1, it is enough to prove that all congestion games are potential games.

Fix a congestion game, and for all strategy profiles $\mathbf{s}$ define

$$\Phi(\mathbf{s}) = \sum_{e\in E} \sum_{j=1}^{f_e(\mathbf{s})} c_e(j). \tag{5}$$

We will show that Φ is a potential function for our congestion game. The particular choice of potential in (5) is known as *Rosenthal's* potential [4].

Fix a strategy profile $\mathbf{s}$, a player i, and assume i deviates to a new set of resources $s_i' \in S_i$. For convenience, for every $e \in E$, denote $f_e = f_e(\mathbf{s})$ and $f_e' = f_e(s_i', \mathbf{s}_{-i})$. Then notice that, given that all other players remain fixed at $\mathbf{s}_{-i}$, for all resources $e \in s_i' \cap s_i$ that player i keeps occupying, and for all resources $e \in E \setminus (s_i \cup s_i')$ that player i does not play in either of her strategies, the congestion remains unchanged, i.e. $f_e' = f_e$. On the other hand, for all "new" resources $e \in s_i' \setminus s_i$ we have that $f_e' = f_e + 1$, and for all "abandoned" resources $e \in s_i \setminus s_i'$ we have $f_e' = f_e - 1$.

Putting all these together, we get that

$$\begin{aligned}
\Phi(s_i', \mathbf{s}_{-i}) - \Phi(\mathbf{s}) &= \sum_{e\in E}\sum_{j=1}^{f_e'} c_e(j) - \sum_{e\in E}\sum_{j=1}^{f_e} c_e(j) \\
&= \sum_{e\in E}\left(\sum_{j=1}^{f_e'} c_e(j) - \sum_{j=1}^{f_e} c_e(j)\right) \\
&= \sum_{e\in s_i'\setminus s_i}\left(\sum_{j=1}^{f_e+1} c_e(j) - \sum_{j=1}^{f_e} c_e(j)\right) + \sum_{e\in s_i\setminus s_i'}\left(\sum_{j=1}^{f_e-1} c_e(j) - \sum_{j=1}^{f_e} c_e(j)\right) \\
&+ \sum_{e\in s_i'\cap s_i}\left(\sum_{j=1}^{f_e} c_e(j) - \sum_{j=1}^{f_e} c_e(j)\right) + \sum_{e\in E\setminus(s_i\cup s_i')}\left(\sum_{j=1}^{f_e} c_e(j) - \sum_{j=1}^{f_e} c_e(j)\right) \\
&= \sum_{e\in s_i'\setminus s_i} c_e(f_e+1) + \sum_{e\in s_i\setminus s_i'} (-c_e(f_e)) + 0 + 0 \\
&= \sum_{e\in s_i'\setminus s_i} c_e(f_e+1) - \sum_{e\in s_i\setminus s_i'} c_e(f_e),
\end{aligned}$$

while

$$\begin{aligned}
C_i(s_i', \mathbf{s}_{-i}) - C_i(\mathbf{s}) &= \sum_{e\in s_i'} c_e(f_e') - \sum_{e\in s_i} c_e(f_e) \\
&= \left(\sum_{e\in s_i'\setminus s_i} c_e(f_e') + \sum_{e\in s_i'\cap s_i} c_e(f_e')\right) - \left(\sum_{e\in s_i\setminus s_i'} c_e(f_e) + \sum_{e\in s_i\cap s_i'} c_e(f_e)\right) \\
&= \sum_{e\in s_i'\setminus s_i} c_e(f_e+1) + \sum_{e\in s_i'\cap s_i} c_e(f_e) - \sum_{e\in s_i\setminus s_i'} c_e(f_e) - \sum_{e\in s_i\cap s_i'} c_e(f_e) \\
&= \sum_{e\in s_i'\setminus s_i} c_e(f_e+1) - \sum_{e\in s_i\setminus s_i'} c_e(f_e),
\end{aligned}$$

thus indeed satisfying definition (1). □

3 Potentials and Price of Stability

Theorem 3. *Assume that Φ is a potential function for (cost-minimization) game $\mathcal{G}$. If there exist constants $\alpha, \beta > 0$ such that, for any strategy profile* $\mathbf{s}$,

$$\alpha \cdot C(\mathbf{s}) \leq \Phi(\mathbf{s}) \leq \beta \cdot C(\mathbf{s}), \tag{6}$$

then the Price of Stability[3] *of $\mathcal{G}$ is at most $\frac{\beta}{\alpha}$.*

Proof. Since the Price of Stability is defined as the cost of the *best* pure Nash equilibrium over the optimum social cost, it is enough if we show that there exists a pure equilibrium $\tilde{\mathbf{s}}$ such that

$$C(\tilde{\mathbf{s}}) \leq \frac{\beta}{\alpha} C(\mathbf{s}^*),$$

where $\mathbf{s}^* \in \operatorname{argmin}_{\mathbf{s}} C(\mathbf{s})$ is a socially optimum profile.

We choose $\tilde{\mathbf{s}} \in \operatorname{argmin}_{\mathbf{s}} \Phi(\mathbf{s})$ to be a minimizer of the potential. We know (from the proof of Theorem 1) that $\tilde{\mathbf{s}}$ is indeed an equilibrium. Furthermore, using (6) we get that

$$\alpha C(\tilde{\mathbf{s}}) \leq \Phi(\tilde{\mathbf{s}}) \leq \Phi(\mathbf{s}^*) \leq \beta C(\mathbf{s}^*),$$

where the middle inequality holds because $\tilde{\mathbf{s}}$ is a potential minimizer. □

Corollary 1. *The Price of Stability of congestion games*[4] *with cost functions from the set $\mathcal{C}_d$ of polynomials*[5] *of maximum degree d, is at most $d + 1$.*

Proof. We know that congestion games are potential games (see Theorem 2). Furthermore, one can show[6] that if all cost functions c_e are from $\mathcal{C}_d$, then Rosenthal's potential (defined in (5)) satisfies condition (6) of Theorem 3 with $\alpha = \frac{1}{d+1}$ and $\beta = 1$. □

It is important to emphasize here that in both of our main positive results of the two last sections, namely Theorems 2 and 3, we didn't make *any* assumption about the monotonicity of the cost functions. That's the reason why, at the start of Section 2 we chose to define congestion games in a general way, allowing for arbitrary costs. On the contrary, when we study selfish routing games, we need to assume nondecreasing cost functions for the edges (which, for example, is essential for the Price of Anarchy upper bound of 5/2 for affine costs[7]) Summarizing, we have the following

Remark 2. *Theorems 2 and 3 hold for congestion games with arbitrary cost functions. In particular, they are valid even for resources with decreasing costs.*

4 Weighted Congestion Games

Weighted congestion games generalize "standard" congestion games (introduced in Section 2) by associating with each player i a positive *weight* $w_i \in \mathbb{R}_{>0}$. Then, the cost of player i in (3) is modified to

$$C_i(\mathbf{s}) = w_i \sum_{e \in s_i} c_e(f_e(\mathbf{s})),$$

[3] With respect to pure Nash equilibria.

[4] And thus, also of atomic selfish routing games (see Remark 1).

[5] Formally, $\mathcal{C}_d = \{x \mapsto \sum_{i=0}^{d} a_i x^i \mid a_i \geq 0 \;\; \forall i = 0, \dots, d\}$.

[6] We leave this as an exercise for your next assignment.

[7] See Eq. (12.7) in the proof of Theorem 12.3 in [20LAGT].

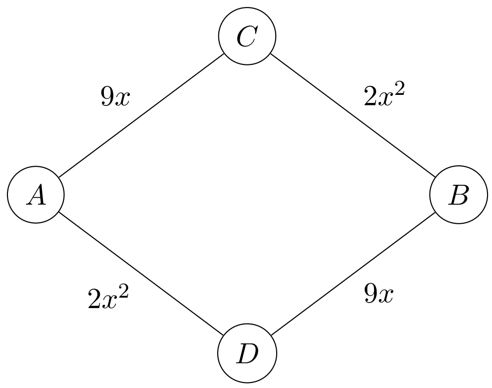

Figure 1: A weighted congestion game with no pure Nash equilibria. Player 1 has weight 1 and want to go from A to B. Player 2 has weight 2 and want to go from C to D.

where $f_e(\mathbf{s}) = \sum_{i:e\in s_i} w_i$ now denotes the sum of the weights of the players that use resource e. You can think of this modification as cost function c_e now representing the congestion on resource e "per unit of weight" that gets assigned to it. Similarly, the total cost in (4) becomes

$$C(\mathbf{s}) = \sum_{i=1}^{n} C_i(\mathbf{s}) = \sum_{i=1}^{n} w_i \sum_{e\in s_i} c_e(f_e(\mathbf{s})) = \sum_{e\in E} f_e(\mathbf{s}) \cdot c_e(f_e(\mathbf{s})).$$

It is easy to see that this is indeed a generalization of unweighted congestion games: just set all player weights equal to $w_i = 1$.

Unfortunately, weighted congestion games fail to share many of the good properties of their unweighted counterparts that we discussed in the previous sections. In particular, in general they are *not* potential games and they don't always have pure Nash equilibria, as the following example demonstrates.

Example 2. Consider the following *weighted* congestion game with $n = 2$ players, inspired from a similar construction by Hansknecht et al. [2]. We have an undirected graph with 4 nodes, namely A, B, C and D, shown in Figure 1. Edges (A, C) and (B, D) have cost function $c_1(x) = 9x$, while edges (B, C) and (A, D) have $c_2(x) = 2x^2$.

Player 1 has weight $w_1 = 1$, and wants to travel from A to B; her feasible strategies are either path $A \to C \to B$ (called *Up* (U) from now on for simplicity), or $A \to D \to B$ (*Down* (D)). Player 2 has weight $w_2 = 2$ and wants to go from C to D; her strategies are either $C \to B \to D$ (*Right* (R)) or $C \to A \to D$ (*Left* (L)).

We will prove that the above game has *no* pure Nash equilibria. Due to symmetry, it is enough to prove that outcomes (U, R) and (D, R) are not equilibria (the remaining profiles (D, L) and (U, L) are essentially identical). More specifically, we will demonstrate the instability of outcome (U, R) by showing that player 2 has an incentive to unilaterally deviate to (U, L). Similarly, for (D, R), player 1 will benefit by deviating to (U, R).

First for (U, R), we compute

$$C_2(U, R) = w_2 \cdot c_2(w_1 + w_2) + w_2 \cdot c_1(w_2) = 2c_2(3) + 2c_1(2) = 2 \cdot 18 + 2 \cdot 18 = 72$$
$$C_2(U, L) = w_2 \cdot c_1(w_1 + w_2) + w_2 \cdot c_2(w_2) = 2c_1(3) + 2c_2(2) = 2 \cdot 27 + 2 \cdot 8 = 70,$$

so indeed $C_2(U, L) < C_2(U, R)$.

Similarly, for (D, R) we have

$$C_1(D, R) = w_1 \cdot c_2(w_1) + w_1 \cdot c_1(w_1 + w_2) = c_2(1) + c_1(3) = 2 + 27 = 29$$
$$C_1(U, R) = w_1 \cdot c_1(w_1) + w_1 \cdot c_2(w_1 + w_2) = c_1(1) + c_2(3) = 9 + 18 = 27.$$

5 Equilibrium Existence in Nonatomic Routing Games

In Theorem 2 we showed that atomic selfish routing games always have equilibria. A similar result holds for nonatomic routing games: an equilibrium flow[8] always exists. But something even stronger holds now; these equilibria are *unique*, in the sense that all equilibrium flows induce the same delay on all edges of the graph (and thus, also the same social cost):

Theorem 4. *Every nonatomic selfish routing game has at least one equilibrium flow. Furthermore, if $\tilde{f}_1, \tilde{f}_2$ are both equilbrium flows, then for any edge e of the routing graph:*

$$c_e(\tilde{f}_1) = c_e(\tilde{f}_2).$$

An immediate result of the above theorem is that the notions of Price of Anarchy and Price of Stability coincide in nonatomic routing games.

A detailed proof of Theorem 4 can be found in [3, Section 18.3.1], here we only sketch the high level idea. Inspired by the potential method described in Theorem 2, and Rosenthal's potential (5) in particular, we would like to construct a continuous analogue. Indeed, if we define function

$$\Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(t)\, dt,$$

over the set of feasible flows f, it turns out that its minimizers correspond *exactly* to the equilibrium flows of the nonatomic routing game. This is a consequence of the fact that the above optimization problem is convex (see, e.g., [1, Chapter 2]).

References

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[2] C. Hansknecht, M. Klimm, and A. Skopalik. Approximate Pure Nash Equilibria in Weighted Congestion Games. In K. Jansen, J. D. P. Rolim, N. R. Devanur, and C. Moore, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2014)*, pages 242–257, 2014. doi: 10.4230/LIPIcs.APPROX-RANDOM.2014.242.

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[4] R. W. Rosenthal. A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, 2(1):65–67, 1973.

[8] For the formal definition of equilibrium flows in nonatomic games, recall Definition 11.3 of your texbook [20LAGT].