



Exercise Set 2

Due Wednesday, May 15th, 11:45

Special Topics in Algorithmic Game Theory (MA5226)

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Bonus System: Students are expected to solve these exercise sets, and they can qualify for a grade bonus by writing down and handing in their answers, either in person or by e-mail to diogo.pocas@tum.de or alexandros.tsigonias@tum.de. Each student that hands in a sufficient amount of satisfactory solutions achieves a grade bonus, which improves the grade of a passed exam by one grade (e.g. a 4.0 becomes a 3.7, or a 3.7 becomes a 3.3, etc. Note that a 1.0 cannot be improved). **No late submissions accepted.**

Exercise 2.1 (*Sponsored search with quality*, Exercise 3.4 from [20LAGT])

Consider the following extension of the sponsored search setting described in class. Each bidder i now has a publicly known *quality* β_i , in addition to a private valuation v_i per click. As usual, each slot j has a CTR α_j , and $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k$. We assume that if bidder i is placed in slot j , then the probability of a click is $\beta_i \alpha_j$. Thus bidder i derives value $v_i \beta_i \alpha_j$ from the j th slot.

Describe the welfare-maximizing allocation rule in this generalized sponsored search setting. Prove that this rule is monotone. Give an explicit formula for the per-click payment of each bidder that extends this allocation rule to a DSIC mechanism.

Exercise 2.2 (*Revenue target auctions*, Problem 3.2 from [20LAGT])

This problem considers a k -unit auction (Exercise 1.5 from last week) in which the seller has a specific revenue target R . Consider the following algorithm that, given bids \mathbf{b} as input, determines the winning bidders and their payments.

Revenue Target Auction

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initialize a set  $S$  to the top  $k$  bidders
while there is a bidder  $i \in S$  with  $b_i < R/|S|$  do
    remove an arbitrary such bidder from  $S$ 
    allocate an item to each bidder of  $S$  (if any), and charge each
    of them a price equal to the maximum between  $R/|S|$  and the
     $(k+1)$ -th highest bid.
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- Give an explicit description of the allocation rule of the Revenue Target Auction, and prove that it is monotone.
- Conclude from Myerson's lemma that the Revenue Target Auction is a DSIC mechanism.
- Prove that whenever the DSIC and welfare-maximizing k -unit auction (Exercise 1.5 from last week) obtains revenue at least R , the Revenue Target Auction obtains revenue R .
- Prove that there exists a valuation profile for which the Revenue Target Auction obtains revenue R but the DSIC and welfare-maximizing auction earns revenue less than R .

Exercise 2.3 (*Single-item auction with two uniform bidders*, Exercise 5.1 from [20LAGT])

Consider a single-item auction with two bidders with valuations drawn independently from the uniform distribution on $[0, 1]$.

- (a) Prove that the expected revenue obtained by a second-price auction (with no reserve) is $\frac{1}{3}$.
- (b) Prove that the expected revenue obtained by a second-price auction with reserve $\frac{1}{2}$ is $\frac{5}{12}$.

Exercise 2.4 (*Virtual valuations*, Exercise 5.2 from [20LAGT])

Compute the virtual valuation function of the following distributions.

- (a) The uniform distribution on $[0, a]$ with $a > 0$.
- (b) The exponential distribution with rate $\lambda > 0$ (on $[0, \infty)$).
- (c) The distribution given by $F(v) = 1 - \frac{1}{(v+1)^c}$ on $[0, \infty)$, where $c > 0$ is some constant.

Exercise 2.5 (*MHR condition and regularity*, Exercises 5.3 and 5.4 from [20LAGT])

A valuation distribution meets the *monotone hazard rate (MHR)* condition if its *hazard rate* $\frac{f_i(v_i)}{1-F_i(v_i)}$ is nondecreasing in v_i .

- (a) Prove that every distribution meeting the MHR condition is regular.
- (b) Which of the distributions in Exercise 2.4 are regular? Which of them satisfy the MHR condition?

Exercise 2.6 (*Distribution violating Theorem 1*, Exercise 5.6 from [20LAGT])

For the valuation distribution in Exercise 2.4(c) with $c = 1$, argue that the expected revenue of an auction does *not* necessarily equal its expected virtual welfare. How do you reconcile this observation with Theorem 1 (Lecture 4 - Supplementary notes)?

Exercise 2.7 (*Bayes-Nash equilibrium and first-price auctions*, Problem 5.3 from [20LAGT])

This problem introduces the Bayes-Nash equilibrium concept and compares the expected revenue earned by first-price and second-price single-item auctions.

First-price auctions have no dominant strategies, and we require a new concept to reason about them. For this problem, assume that bidders' valuations are drawn i.i.d. from a commonly known distribution F . A *strategy* of a bidder i in a first-price auction is a predetermined plan for bidding — a function $b_i(\cdot)$ that maps her valuation v_i to a bid $b_i(v_i)$. The semantics are: "when my valuation is v_i , I will bid $b_i(v_i)$." We assume that bidders' strategies are common knowledge, with bidders' valuations (and hence induced bids) private as usual.

A strategy profile $b_1(\cdot), \dots, b_n(\cdot)$ is a *Bayes-Nash equilibrium* if every bidder always bids optimally given her information — if for every bidder i and every valuation v_i , the bid $b_i(v_i)$ maximizes i 's expected utility, where the expectation is with respect to the distribution over others bids induced by F and \mathbf{b}_{-i} .

- (a) Suppose F is the uniform distribution on $[0, 1]$. Verify that setting $b_i(v_i) = v_i(n - 1)/n$ for every i and v_i yields a Bayes-Nash equilibrium.
- (b) Prove that the expected revenue of the seller (over \mathbf{v}) at this equilibrium of the first-price auction is exactly the expected revenue of the seller in the truthful outcome of a second-price auction.
- (c) Extend the conclusion in (b) to every continuous and strictly increasing distribution function F on $[0, 1]$.

This problem set will be discussed in the tutorials on May 17th/22nd, 2019.