



Exercise Set 3

Due Wednesday, May 22nd, 12:15

Special Topics in Algorithmic Game Theory (MA5226)

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Hand in your answers, either in person or by e-mail to diogo.pocas@tum.de or alexandros.tsigonias@tum.de. **No late submissions accepted.**

Exercise 3.1 (*Optimal single-item auction with independent bidders*, Exercise 6.1 from [20LAGT])

Consider an n -bidder single-item auction, with bidders' valuations drawn independently from regular distributions F_1, \dots, F_n .

- Give a formula for the winner's payment in an optimal auction, in terms of the bidders' virtual valuation functions.
- Show by example that, in an optimal auction, the highest bidder need not win, even if they have a positive virtual valuation.
- Give an intuitive explanation of why the property in (b) might be beneficial to the expected revenue of an auction.

Exercise 3.2 (*Second-price auction with bidder-specific reserve prices*, Exercise 6.3 from [20LAGT])

Prove that with regular valuation distributions F_1, \dots, F_n , the allocation rule of a second-price auction with bidder-specific reserve prices $r_i = \varphi_1^{-1}(t)$ is monotone.

Exercise 3.3 (*Optimal k -unit auction*, Exercise 5.7 from [20LAGT])

Consider a k -unit auction (Exercises 1.5 and 2.2 from previous weeks) in which bidders' valuations are drawn i.i.d. from a regular distribution F . Describe an optimal auction. Which of the following does the reserve price depend on: k , n , or F ?

Exercise 3.4 (*Comparing expected revenues with regular bidders*, Exercise 6.4 from [20LAGT])

Consider an n -bidder single-item auction, with bidders' valuations drawn i.i.d. from a regular distribution F . Prove that the expected revenue of a second-price auction (with no reserve price) is at least $\frac{n-1}{n}$ times that of an optimal auction.

Exercise 3.5 (*Regular distributions: concavity and median pricing*, related to Problems 5.1 and 5.2 from [20LAGT])

This problem derives an interesting interpretation of a virtual valuation $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ and the regularity condition. Consider a strictly increasing distribution function F with a strictly positive density function f on the interval $[0, v_{\max}]$, with $v_{\max} < +\infty$.

For a single bidder with valuation drawn from F , for $q \in [0, 1]$, define $V(q) = F^{-1}(1 - q)$ as the (unique) posted price that yields a probability q of a sale. Define $R(q) = q \cdot V(q)$ as the expected revenue obtained from a single bidder when the probability of a sale is q . The function $R(q)$, for $q \in [0, 1]$, is the *revenue curve* of F . Note that $R(0) = R(1) = 0$.

- What is the revenue curve for the uniform distribution on $[0, 1]$?
- Prove that the revenue curve of a regular distribution is concave.
- Fix now a single-item, single-bidder environment with valuation drawn from a regular distribution F . Prove that offering to the bidder a price equal to the median of F (i.e. the value for which $F(p) = \frac{1}{2}$), guarantees at least 50% of the optimal revenue (in expectation).

- (d) Show that, under the same conditions as (c), the optimal revenue is upper bounded by the median of F .

Exercise 3.6 (*Improving the prophet inequality*, related to Problem 6.1 from [20LAGT])

This problem investigates improvements to the prophet inequality (Theorem 6.1 from [20LAGT]).

- (a) Show that the factor of $\frac{1}{2}$ in the prophet inequality cannot be improved, even for $n = 2$: for every constant $c > \frac{1}{2}$, there are distributions^a G_1, G_2 such that *every* strategy, threshold or otherwise, has expected value less than $c \cdot \mathbf{E}_{\pi \sim \mathbf{G}} [\max_i \pi_i]$.
- (b) Can the factor of $\frac{1}{2}$ in the prophet inequality be improved for the special case of i.i.d. distributions, with $G_1 = \dots = G_n$?

^aRecall that the prophet inequality does *not* necessarily assume continuity of the probability distributions. Thus, if you like, feel free to use *discrete* distributions in this exercise.

This problem set will be discussed in the tutorials on May 24th/29th, 2019.