Chair of Operations Research
Department of Mathematics
TU Munich

Exercise Set 4

Due Wednesday, May 29th, 11:45

Special Topics in Algorithmic Game Theory (MA5226)

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Hand in your answers, either in person or by e-mail to `diogo.pocas@tum.de` or `alexandros.tsigonias@tum.de`. **No late submissions accepted.**

Exercise 4.1 (*Computation of Pigou bounds*, Exercises 11.1 and 11.2 from [20LAGT])

(a) Prove that if $\mathcal{C}$ is the set of cost functions of the form $c(x) = ax + b$ with $a, b \geq 0$, then the Pigou bound $\alpha(\mathcal{C}) = \frac{4}{3}$.

(b) Prove that if $\mathcal{C}$ is the set of nonnegative, nondecreasing, and concave cost functions, then the Pigou bound $\alpha(\mathcal{C}) = \frac{4}{3}$.

Exercise 4.2 (*Pigou network with monomial cost function*)
For any positive integer p, consider the (nonlinear) Pigou network with one unit of incoming traffic and cost functions $c_1(x) = 1$, $c_2(x) = x^p$. Show that the Price of Anarchy is $\Omega\left(\frac{p}{\log p}\right)$.

(*Hint: you can use without proof the approximation* $1 \leq \sqrt[p]{p+1} \leq 1 + 2\frac{\ln(p+1)}{p}$*, valid for any* $p \geq 1$.)

Exercise 4.3 (*Price of anarchy for maximum travel time objective*, Problem 11.2 from [20LAGT])
In this problem we consider an alternative objective function, that of minimizing the *maximum* travel time of a flow f,

$$\max_{P \in \mathcal{P}: f_P > 0} \sum_{e \in P} c_e(f_e).$$

The price of anarchy (PoA) with respect to this objective is then defined as the ratio between the maximum cost of an equilibrium flow and that of a flow with minimum-possible maximum cost.

We assume throughout this problem that there is one origin, one destination, one unit of traffic, and affine cost functions (of the form $c_e(x) = a_e x + b_e$ for $a_e, b_e \geq 0$).

(a) Prove that in networks with only two vertices o and d, and any number of parallel edges, the PoA with respect to the maximum cost objective is 1.

(b) Prove that the PoA with respect to the maximum cost objective can be as large as 4/3.

(c) Prove that the PoA with respect to the maximum cost objective is never larger than 4/3.

Exercise 4.4 (*Two-hop path Nash equilibrium*, Exercise 12.5 from [20LAGT])
Recall the four-agent atomic selfish routing network in Figure 12.3 from [20LAGT]. Verify that if each agent routes her traffic on her two-hop path, then the result is an equilibrium flow.

Exercise 4.5 (*Price of Anarchy with two agents*, Problem 12.2(b) from [20LAGT])
Recall the four-agent atomic selfish routing network in Figure 12.3 from [20LAGT], where the Price of Anarchy is 2.5. How large can the Price of Anarchy be with affine cost functions and only *two* agents?

Exercise 4.6 (*Price of Anarchy of scheduling games*, Problem 12.3 from [20LAGT])

This problem studies a scenario with k agents, where agent i has a positive weight w_i. There are m identical machines. Each agent chooses a machine, and wants to minimize the *load* of her machine, defined as the sum of the weights of the agents who choose it. This problem considers the objective of minimizing the *makespan*, defined as the maximum load of a machine. A *pure Nash equilibrium* is an assignment of agents to machines so that no agent can unilaterally switch machines and decrease the load she experiences.

(a) Prove that the makespan of a pure Nash equilibrium is at most twice that of the minimum possible.

(b) Prove that, as k and m tend to infinity, pure Nash equilibria can have makespan arbitrarily close to twice the minimum possible.

This problem set will be discussed in the tutorials on May 31st/ June 5th, 2019.