Chair of Operations Research
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Exercise Set 5

Due Wednesday, June 5th, 11:45

Special Topics in Algorithmic Game Theory (MA5226)

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Hand in your answers, either in person or by e-mail to `diogo.pocas@tum.de` or `alexandros.tsigonias@tum.de`. **No late submissions accepted.**

Exercise 5.1 (*Lower bound on Price of Stability*)
Show that the Price of Stability of the class of atomic selfish routing networks with cost functions from the family $\mathcal{C}_d$ of polynomials with maximum degree d (and nonnegative coefficients) is $\Omega(\frac{d}{\log d})$, i.e. at least the value of the Pigou bound $\alpha(\mathcal{C}_d)$.

Exercise 5.2 (*Relating potential and cost functions*, related to Exercise 13.4 and Problem 15.3 from [20LAGT])
Consider the class of congestion games with cost functions from the family $\mathcal{C}_d$ of polynomials with maximum degree d (and nonnegative coefficients). Prove that, for any strategy profile $\mathbf{s}$,

$$\frac{1}{d+1}C(\mathbf{s}) \leq \Phi(\mathbf{s}) \leq C(\mathbf{s}),$$

where $C(\mathbf{s})$ is the social cost and $\Phi(\mathbf{s})$ is Rosenthal's potential.[a]

Exercise 5.3 (*Weighted congestion games with affine cost functions have pure equilibria*, related to Exercise 13.6 from [20LAGT])
We saw in class[b] that, in general, weighted congestion games may not have pure Nash equilibria. However, for the special case when all resources have affine cost functions, they actually do have. Prove this, by using the following function as a potential:

$$\Phi(\mathbf{s}) = \sum_{e \in E} \left(c_e(f_e(\mathbf{s}))f_e(\mathbf{s}) + \sum_{i:e \in s_i} c_e(w_i)w_i \right).$$

Exercise 5.4 (*Pure strategies in the support of a mixed Nash equilibrium*, related to Exercise 13.1 from [20LAGT])

(a) Prove that allowing deviations to mixed strategies does not change the notion of mixed Nash equilibrium. Formally, if $\sigma = \sigma_1 \times \ldots \times \sigma_n$ is a mixed Nash equilibrium (as in Definition 13.3 from [20LAGT]) and σ_i' is a *mixed* strategy of player i, then

$$\mathbf{E}_{\mathbf{s}\sim\sigma}\left[C_i(\mathbf{s})\right] \leq \mathbf{E}_{s_i'\sim\sigma_i',\mathbf{s}_{-i}\sim\sigma_{-i}}\left[C_i(s_i', \mathbf{s}_{-i})\right].$$

(b) Show that, at any mixed Nash equilibrium, all pure strategies in the support of any player induce the same expected cost. Formally, if $\sigma = \sigma_1 \times \ldots \times \sigma_n$ is a mixed Nash equilibrium and σ_i assigns strictly positive probabilities to both (pure) strategies s_i, s_i', then

$$\mathbf{E}_{\mathbf{s}_{-i}\sim\sigma_{-i}}\left[C_i(s_i, \mathbf{s}_{-i})\right] = \mathbf{E}_{\mathbf{s}_{-i}\sim\sigma_{-i}}\left[C_i(s_i', \mathbf{s}_{-i})\right].$$

[a] See Equations (4) and (5), respectively, from Lecture's 8 Supplementary Notes.
[b] See Section 4 from Lecture's 8 Supplementary Notes.

Exercise 5.5 (*Equilibrium concepts for the traffic light game*)

Recall the traffic light game seen in class with two agents, each having one of two possible actions ('Stop' or 'Go').[c] We have seen that

- ('Stop','Go') and ('Go','Stop') are pure Nash equilibria;
- there is a mixed Nash equilibrium where each agent plays 'Go' with probability $\frac{1}{5}$;
- there is a correlated equilibrium which randomizes uniformly between the two pure Nash equilibria.

(a) Are there any other mixed Nash equilibria? (prove or give an example)

(b) Are there any coarse correlated equilibria which are *not* correlated equilibria? (prove or give an example)

(c) Show that the proper[d] mixed Nash equilibrium given in class achieves the *worst* expected social cost among all coarse correlated equilibria.

(*Hint: feel free to use any LP solver you wish, in any programming language you like, such as Mathematica, Matlab, Python,* `http://www.phpsimplex.com/en/`*, etc.*)

Exercise 5.6 (*Price of Anarchy of mixed Nash equilibria for scheduling games*, Problem 13.1 in [20LAGT])

Recall the class of cost-minimization games introduced in Exercise 4.6 from last week (also Problem 12.3 in [20LAGT]), where each agent $i = 1, 2, \ldots, k$ has a positive weight w_i and chooses one of m identical machines to minimize her load. We again consider the objective of minimizing the makespan, defined as the maximum load of a machine. Prove that, as k and m tend to infinity, the worst-case Price of Anarchy of *mixed* Nash equilibria in such games is *not* upper bounded by any constant.

(*Hint: you can invoke well-known properties of 'balls-into-bins' occupancy problems.*)

[c] Its cost matrix can be found in page 177 of your textbook [20LAGT].

[d] That is, the mixed Nash equilibrium which is *not* pure.

This problem set will be discussed in the tutorials on June 12th, 2019.