Special Topics in Algorithmic Game Theory
Lecture 3 – Supplementary Notes

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1 Myerson's Lemma

Recall from the class lecture that a (direct-revelation) *mechanism* $(\mathbf{x}, \mathbf{p})$ (in a single-parameter environment with n agents and feasibility set $X \subseteq \mathbb{R}^n_{\geq 0}$) comprises of (a) an *allocation rule* $\mathbf{x} = (x_1, \dots, x_n) : \mathbb{R}^n_{\geq 0} \longrightarrow X$ and (b) a *payment rule* $\mathbf{p} = (p_1, \dots, p_n) : \mathbb{R}^n_{\geq 0} \longrightarrow \mathbb{R}^n_{\geq 0}$. The semantics here are the following

1. The agents report a *bid profile* $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n_{\geq 0}$ to the mechanism.
2. Based on this input, the mechanism computes $\mathbf{x}(\mathbf{b})$ and $\mathbf{p}(\mathbf{b})$.
3. Each agent $i = 1, \dots, n$ gets allocated $x_i(\mathbf{b})$ "units" and pays $p_i(\mathbf{b})$.

The property that $p_i(\mathbf{b}) \geq 0$ for all players i and bids $\mathbf{b}$ is sometimes called *No Positive Transfers (NPT)*, to capture the fact that we do not allow our mechanisms to send money *to* the agents but only receive payments *from* them.

An allocation rule $\mathbf{x}$ is called *monotone* if, for every player i and any bids $\mathbf{b}_{-i}$ from the remaining agents, $x_i(z, \mathbf{b}_{-i})$ is a nondecreasing function of z. Also, an allocation $\mathbf{x}$ is called *implementable* (in dominant strategies) if there is a payment rule $\mathbf{p}$ such that $(\mathbf{x}, \mathbf{p})$ is a DSIC mechanism.

Theorem 1 (Myerson's Lemma [2])**.** *A (direct-revelation) mechanism $(\mathbf{x}, \mathbf{p})$ is DSIC, if and only if*

1. *its allocation rule $\mathbf{x}$ is monotone, and*
2. *the payment charged to player i is given by the formula*

$$p_i(\mathbf{b}) = b_i \cdot x_i(\mathbf{b}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i})\, dz. \tag{1}$$

In particular, a consequence of this theorem is that an allocation rule is implementable if and only if it is monotone.

Proof. ($\Longrightarrow$) Fix some DSIC mechanism $(\mathbf{x}, \mathbf{p})$. Also fix some player i, and all other players' bids $\mathbf{v}_{-i}$. For simplicity, let's denote $x(z) \equiv x_i(z, \mathbf{v}_{-i})$ and $p(z) \equiv p_i(z, \mathbf{v}_{-i})$. Also, denote with $u(z)$ the utility of player i when she truthfully bids z, i.e.,

$$u(z) = x(z) \cdot z - p(z). \tag{2}$$

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Then, due to DSIC, it must be that

$$x(z)z - p(z) = u(z) \geq x(y)z - p(y) \tag{3}$$
$$x(y)y - p(y) = u(y) \geq x(z)y - p(z). \tag{4}$$

The first inequality simply corresponds to the fact that when player i has true value z, his utility is always maximized when truthfully reporting z, instead of any other value y. The second inequality corresponds to the symmetric situation, by switching variables z and y. Subtracting (3) from (4) we get that

$$[x(z)y - p(z)] - [x(z)z - p(z)] \leq u(y) - u(z) \leq [x(y)y - p(y)] - [x(y)z - p(y)],$$

or equivalently,

$$x(z)(y - z) \leq u(y) - u(z) \leq x(y)(y - z). \tag{5}$$

From this we get that

$$z < y \quad \Longrightarrow \quad y - z > 0 \quad \Longrightarrow \quad x(z) \leq \frac{u(y) - u(z)}{y - z} \leq x(y), \tag{6}$$

which shows that indeed $x_i(z, \mathbf{v}_{-i})$ is nondecreasing with respect to player i's report z.

To show now that the payment rule $p(z)$ is given by (1), observe (5): it says that[1] $x(z)$ is a *subgradient*[2] of function u at z, and this holds for all possible bids z. As a result, the utility function $u(z)$ is an absolutely continuous function, with the Fundamental Theorem of Calculus giving that

$$u(b) = u(0) + \int_0^b x(z)\, dz. \tag{7}$$

Furthermore, recall that for all z it must be that $p(z) \geq 0$ (due to NPT) and $u(z) \geq 0$ (since our mechanism is DSIC, and thus has to satisfy Individual Rationality[3] (IR)). So,

$$0 \leq u(0) = x(0) \cdot 0 - p(0) = -p(0) \leq 0,$$

meaning that $u(0) = 0$, and thus (7) finally becomes

$$x(b)b - p(b) = \int_0^b x(z)\, dz,$$

which is exactly the desired expression (1).

($\Longleftarrow$) For the inverse direction now, assume that an allocation rule $\mathbf{x}$ is monotone and that its payment rule $\mathbf{p}$ is given by (1). We will show that mechanism $(\mathbf{x}, \mathbf{p})$ is DSIC. Fix some player i and the bids of all other players $\mathbf{b}_{-i}$. Also fix a *true* valuation $v \equiv v_i$ for player i. Again, for simplicity, we drop the subscripts i from now on. The utility of player i when she reports bid b is

$$x(b) \cdot v - p(b) = x(b)v - \left[x(b)b - \int_0^b x(z)\, dz\right] = x(b) \cdot (v - b) + \int_0^b x(z)\, dz.$$

[1] As a matter of fact, just the left-hand side inequality is enough for this.

[2] For more details see, e.g., Proposition (11) in Karlin and Peres [1, Appendix C]. Slightly sacrificing rigorousness here, you can think of this in the following way: taking the limit $y \to z$ in (6), and assuming continuity of the allocation function $x(\cdot)$ at at z, you can see that $x(z)$ is equal to the derivative $u'(z)$ of $u(\cdot)$ at z; the allocation of a player is simply the derivative of her utility! However, $x(z)$ might not be continuous at all points z, and thus, $u(z)$ might not be differentiable. Nevertheless, since $x(z)$ is monotone, we know that such "problematic" behaviour can only occur on a set of points with zero (Lebesgue) measure. More specifically, $x(z)$ can have only "jump" discontinuities, at most at countably many points z, and thus it is Riemann integrable (see, e.g., Rudin [3, Chapter 4]); in particular, (7) is well-defined.

[3] See Definition 2.3 in your textbook [20LAGT].

And when she truthfully reports $b = v$, it is

$$x(v) \cdot v - p(v) = \int_0^v x(z)\,dz = \int_0^b x(z)\,dz + \int_b^v x(z)\,dz \geq \int_0^b x(z)\,dz + x(b) \cdot (v - b),$$

the last inequality holding due to the fact that $x(z)$ is nondecreasing. Thus, misreporting a general bid b that might not be equal to v, can only give player i less utility. Furthermore, by truthfully reporting $b = v$, player i always gets a nonnegative utility $\int_0^v x(z)\,dz \geq 0$, satisfying IR. The above two properties demonstrate that indeed our mechanism is DSIC.

See also Figure 14.6 in Karlin and Peres [1] for a nice graphical representation of this proof. □

Your textbook [20LAGT] gives two alternative formulas for the payment rule (1) above, under further special assumptions on the allocation function $x_i(z) \equiv x_i(z, \mathbf{b}_{-i})$ of player i.

In general, we only know that this function is nondecreasing with respect to player i's bid z (keeping everything else $\mathbf{b}_{-i}$ fixed), due to Property 1 of Theorem 1. However, by assuming that it is additionally *piecewise constant*[4], we get that

$$p_i(\mathbf{b}) = \sum_{j=1}^{\ell} z_j \cdot [\text{jump of } x_i(\cdot, \mathbf{b}_{-i}) \text{ at } z_j]\,, \tag{8}$$

where $z_1, z_2, \ldots, z_\ell$ are the breakpoints of the allocation function $x_i(z)$ as player's i bid z ranges within $[0, b_i]$.

Alternatively, if we know that $x_i(z)$ is *continuously differentiable*[5] (with derivative $x_i'(z) = \frac{\partial x_i(z, \mathbf{b}_{-i})}{\partial z}$), then we have that

$$p_i(\mathbf{b}) = \int_0^{b_i} z \cdot x_i'(z)\,dz. \tag{9}$$

It is not difficult to see that (8) and (9) are special cases of the general formula (1). Indeed, first for the case where $x_i(z)$ is piecewise constant, observe that the quantity at the right-hand side of (8) is just the area *above* the graph of $x_i(z)$ *inside* the box $[0, b_i] \times [0, x_i(b_i)]$ of the Euclidean plane $\mathbb{R}^2$; thus, it must equal the total area of that box, which is $b_i \cdot x_i(b_i)$, *minus* the area *below* the graph, which is $\int_0^{b_i} x_i(z)\,dz$. This difference, is exactly the right-hand part of (1).

Next, for the case where $x_i(z)$ is continuously differentiable, by using integration by parts, we can rewrite the integral in the right-hand part of (9) as

$$\begin{aligned}
\int_0^{b_i} z \cdot x_i'(z)\,dz &= [z \cdot x_i(z)]_{z=0}^{z=b_i} - \int_0^{b_i} (z)' \cdot x_i(z)\,dz \\
&= b_i \cdot x_i(b_i) - 0 \cdot x_i(0) - \int_0^{b_i} x_i(z)\,dz \\
&= b_i \cdot x_i(b_i) - \int_0^{b_i} x_i(z)\,dz,
\end{aligned}$$

which is exactly the right-hand side of (1).

[4]This is the case, e.g., with the single-item and sponsored search auction examples we have seen before in class.

[5]As a matter of fact, a weaker requirement on $x_i(z)$ is sufficient here, namely that of *absolute continuity*.

References

[1] A. R. Karlin and Y. Peres. *Game Theory, Alive.* American Mathematical Society, 2017. ISBN 9781470419820. URL https://homes.cs.washington.edu/~karlin/GameTheoryBook.pdf.

[2] R. B. Myerson. Optimal Auction Design. *Mathematics of Operations Research*, 6(1):58–73, 1981. doi: 10.1287/moor.6.1.58.

[3] W. Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, 3rd edition, 1976.