Special Topics in Algorithmic Game Theory
Lecture 7 – Supplementary Notes

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1 Atomic Selfish Routing

An *atomic* selfish routing network comprises of

- A directed graph $G = (V, E)$
- For each edge $e \in E$, a nondecreasing cost function $c_e : \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0}$, representing the time needed to travel through it, as a function of the number of players that use it.
- A (finite) set of players $i = 1, 2, \dots, k$, each one of them having a special pair of nodes $o_i, d_i \in V$, (*origin* and *destination*, respectively).

This induces, in a natural way, an atomic selfish routing *game* where

- The strategy set $\mathcal{P}_i$ of each player i comprises of the different ways to travel from her origin to her destination, i.e.

$$\mathcal{P}_i = \{P_i \mid P_i \text{ is an } o_i \to d_i \text{ path in } G\}$$

- Given a strategy profile $\mathbf{P} = (P_1, \dots, P_k) \in \mathcal{P}_1 \times \dots \times \mathcal{P}_k$, the *cost* of player i is the total time needed for her to travel through G (on her selected path P_i), that is

$$C_i(\mathbf{P}) = \sum_{e \in P_i} c_e(f_e(\mathbf{P})),$$

  where $f_e(\mathbf{P}) = |\{i \mid e \in P_i\}|$ denotes the number of players that use edge e.

Some times, we refer to strategy profiles of these games as *flows*, and denote $f \equiv \mathbf{P} = (P_1, \dots, P_k)$ and $f_e \equiv f_e(\mathbf{P})$. For example, this is what your texbook [20LAGT] does; this helps point out the similarities with the *nonatomic* selfish routing model we studied in the previous lecture and, furthermore, can many times simplify the notation.

The total *cost of a flow* f (also called *social cost*) is the total travel time experienced by all players, i.e.

$$C(f) = \sum_{i=1}^{k} C_i(f) = \sum_{i=1}^{k} \sum_{e \in P_i} c_e(f_e) = \sum_{e \in E} f_e \cdot c_e(f_e).$$

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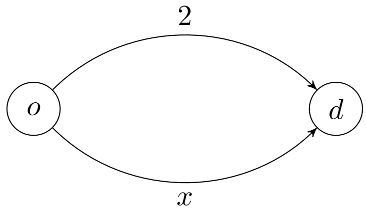

Figure 1: Atomic Pigou-like network with two equilibria. There are two players, both of which want to travel from o to d. The optimum (which is also a equilibrium, and thus the Price of Stability is 1) is to send them from different paths, for a total cost of $2 + 1 = 3$. Going both through the bottom link, for a total cost of $2 \cdot 2 = 4$, is also an equilibrium, resulting in a Price of Anarchy of 4/3.

Definition 1 (Atomic Equilibrium Flow)**.** A flow $f = \mathbf{P}$ (of an atomic selfish routing network) is called an *equilibrium*, if it is a pure Nash equilibrium of the induced game; that is, for every player i and all paths $P_i' \in \mathcal{P}_i$,

$$C_i(\mathbf{P}) \leq C_i(P_i', \mathbf{P}_{-i}).$$

This can be equivalently written as

$$\sum_{e \in P_i} c_e(f_e) \leq \sum_{e \in P_i' \cap P_i} c_e(f_e) + \sum_{e \in P_i' \setminus P_i} c_e(f_e + 1).$$

The equivalence of the two expressions above comes from the following observation: Assume player i deviates from her path P_i to a different one P_i', while all other players remain fixed at their original paths $\mathbf{P}_{-i}$. Then, at the edges of P_i' that already existed in P_i, player i is experiencing exactly the same delay as before (the number of players using them has not changed); at the new edges of P_i', that is, the edges that player i was not using before, but does use now, the congestion has increased by 1 player, namely player i herself.

As we will see in the following lecture, like in the nonatomic case, equilibria *always* exist in atomic selfish routing games as well. However, as the Pigou-like network example of Figure 1 demonstrates[1], we may not have *uniqueness*; that is, an atomic routing game can have many different equilibrium flows. So, in order to adapt the *Price of Anarchy* notion we introduced for nonatomic games, we need to make a selection; we do that in a worst-case analysis approach, choosing the maximum-cost equilibrium:

$$\text{PoA} = \frac{\text{cost of worst equilibrium}}{\text{cost of optimal flow}} = \frac{\max_{\text{equilibria } f} C(f)}{\min_f C(f)}.$$

For the routing game of Figure 1, this ratio is 4/3, equal to the Pigou-bound of the nonatomic case for affine cost function. Notice though that, in addition to the "bad" equilibrium that gives rise to the 4/3 PoA, there is another equilibrium: the optimum flow itself. So, if we were to compare the cost of this equilibrium to the optimal cost, we would get an optimal ratio of 1. As a matter of fact, this notion is also very important in Algorithmic Game Theory, and it's called the *Price of Stability (PoS)*:

$$\text{PoS} = \frac{\text{cost of best equilibrium}}{\text{cost of optimal flow}} = \frac{\min_{\text{equilibria } f} C(f)}{\min_f C(f)}.$$

However, a small adaptation of the Pigou-like network discussed above, gets rid of the aforementioned "good" equilibrium, and results in a game with a *unique* equilibrium, having

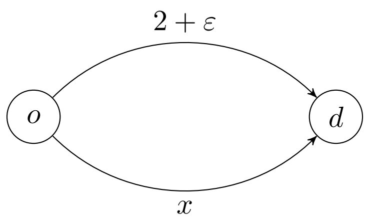

Figure 2: Atomic Pigou-like network with a unique equilibrium. This is a slight variation of the network in Figure 1, with an arbitrarily small $\varepsilon > 0$. The optimum (which is *not* an equilibrium any more) is to send them from different paths, for a total cost of $(2+\varepsilon)+1 = 3+\varepsilon$. Going both through the bottom link, for a total cost of $2 \cdot 2 = 4$, is the *only* equilibrium of the game. This results in $\mathrm{PoA} = \mathrm{PoS} = 4/(3+\varepsilon) \to 4/3$ as $\varepsilon \to 0$.

$\mathrm{PoA} = \mathrm{PoS} = 4/3$. See Figure 2. This is not a coincidence and one can show that, e.g., for polynomial cost functions, the Price of Stability of atomic games is greater or equal to the Price of Anarchy of nonatomic ones[2]. Formally

Proposition 1. *The Price of Stability of the class of atomic selfish routing games with cost functions from the family $\mathcal{C}_d$ of polynomials*[3] *with maximum degree d is $\Omega(d/\log d)$, at least the value of the Pigou-bound $\alpha(\mathcal{C}_d)$.*

[1] Discussed in Figure 12.2 of your textbook [20LAGT].
[2] This will be an exercise in your next assignment.
[3] Formally, $\mathcal{C}_d = \{x \mapsto \sum_{i=0}^{d} a_i x^i \mid a_i \geq 0 \ \ \forall i = 0, \dots, d\}$